

Scalar Field in Any Dimension From the Higher Spin Gauge Theory Perspective

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Abstract

We formulate the equations of motion of a free scalar field in the flat and AdS space of an arbitrary dimension in the form of some “higher spin” covariant constancy conditions. Klein-Gordon equation is interpreted as a non-trivial cohomology of a certain “ σ_- -complex”. The action principle for a scalar field is formulated in terms of the “higher-spin” covariant derivatives for an arbitrary mass in AdS_d and for a non-zero mass in the flat space. The constructed action is shown to be equivalent to the standard first-order Klein-Gordon action at the quadratic level but becomes different at the interaction level because of the presence of an infinite set of auxiliary fields which do not contribute at the free level. The example of Yang-Mills current interaction is considered in some detail. It is shown in particular how the proposed action generates the pseudolocally exact form of the matter currents in AdS_d .

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1 Introduction

Mikhail Vladimirovich Saveliev was a brilliant scientist who made a fundamental contribution to the theory of integrable systems. Misha was always open to a scientific discussion and ready to share his knowledge to colleagues. Working mainly on two-dimensional integrable systems, during last years Saveliev was interested in the application of his ideas to the study of higher-dimensional relativistic supersymmetric models [1] which, he believed, have a chance to be solvable in one or another sense. Now it is an open problem to clarify to which extend his expectations were true.

The aim of this contribution is to reformulate the simplest relativistic model of a scalar field in any dimension in a way inspired by the theory of higher spin gauge fields. So far, the higher spin gauge theory has been developed beyond the linearized level mainly for $d \leq 4$ (see [2, 3] and references therein). From these particular cases it is known that higher spin gauge theories are based on appropriate infinite-dimensional symmetries, higher spin symmetries. All massless fields with spins $s \geq 1$ are gauge fields. In the framework of the formalism developed in [4], the $d = 4$ dynamical equations have a form of some zero-curvature and covariant constancy conditions supplemented with certain constraints. It remains to be analyzed whether this is a signal of some sort of integrability of the 4d higher spin models. The reformulation of the scalar field dynamics suggested in this paper is useful as a starting point towards yet unknown infinite-dimensional higher spin symmetries in any dimension. Also, it sheds some light on the specificities of the higher spin interactions.

The action principle compatible with the higher spin gauge symmetries and general covariance to the lowest nontrivial order in interaction was formulated in AdS_4 [5], thus solving the problem of introducing consistent gravitational interactions of higher spin gauge fields to the cubic order in interactions. The cubic action of [5] was however known to be incomplete requiring some further modification at the higher orders in interactions. Indeed, it is well known that the consistency of some interactions at the cubic level does not guarantee that the theory can be consistently extended beyond the cubic order. (For example, at the cubic level one can consider any number N of spin $3/2$ gravitinos interacting with gravitons but only for $N \leq 8$ it is possible to proceed beyond the cubic level with very specific sets of fields, the supergravitational supermultiplets, carrying spins $s \leq 2$. The true spectrum of fields can only be fixed at the quartic level. The higher spin interactions beyond the cubic order were studied in [6, 4] at the level of equations of motion. From these results and also from the analysis of the unitary lowest weight representations of the higher spin algebras (i.e. higher-spin multiplets) in [7] it is known that the full spectra of spins in the complete higher spin theories contain lower spin fields with spins 1 and $1/2$ and 0. One of the aims of this paper is to construct a spin-0 action in the form analogous to the spin $s > 1$ actions used in [5] as a step towards a complete higher spin action. As argued in [5] unbroken higher spin symmetries require AdS geometry rather than the flat one. We therefore consider the problem both in the flat and AdS background.

Infinite-dimensional higher spin symmetries mix higher derivatives of all orders of the dynamical fields. To make these symmetries manifest it is useful to reformulate dynamics in terms of appropriate higher spin covariant derivatives $\mathcal{D}C^A(x)$. The representation

$C^A(x)$ of the higher spin symmetry is infinite-dimensional containing infinitely many field components (index A), most of which express via the higher derivatives of the dynamical fields by virtue of certain constraints. We formulate the action principle for a free scalar field in terms of the covariant derivative $\mathcal{D}C$ in AdS_d for an arbitrary value of mass m and in the flat space for $m \neq 0$. At the free field level the constructed action is equivalent to the standard first-order Klein-Gordon action because higher components of the representation C^A do not contribute to the action at the quadratic level thus guaranteeing it to be of the normal order in derivatives. However, the proposed action differs from the standard one at the interaction level. In particular, in this framework minimal Yang-Mills interaction of a scalar field turns out to be combined with some additional interactions to the Yang-Mills field strength containing infinite series in higher derivatives of the scalar field with the coefficients proportional to negative powers of the parameter of mass or the cosmological constant. An immediate consequence of our formulation is that it shows how the Yang-Mills current built from the scalar matter can be compensated by a pseudolocal field redefinition, which result provides a generalization to an arbitrary dimension of the observation made for 3d higher spin models [8] that conserved currents are pseudolocally exact in AdS_3 . Also let us mention some parallelism between our results and the description of the consistent interaction of massive higher spin fields with gravity in the recent papers [9, 10], which requires infinite expansions in negative powers in the parameter of mass at the action level. Needless to say that this picture is reminiscent of the α' expansion for massive modes in the string theory.

The paper is organized as follows. In the Section 2, we formulate the equations of motion for a scalar field of an arbitrary mass in the flat space in the “unfolded” form of certain covariant constancy conditions. These results are generalized to AdS_d in the Section 3. In the Section 4, the Klein-Gordon equations are interpreted in terms of the σ_- -cohomology. In the Section 5, we derive the scalar field action formulated in terms of the “higher spin” covariant derivatives of the Sections 2 and 3. The Fock space notation are introduced in the Section 6. Specificities of the Yang-Mills current interaction of the scalar fields described by the action of the Section 5 are discussed in the Section 7. In the Section 8 we generalize the covariant constancy conditions of the sections 2 and 3 to the off-mass-shell system. The Section 9 contains some conclusions.

2 “Unfolded” Formulation of the Scalar Field Equations in the Flat Space

To describe a free massless scalar field $c(x)$ in d dimensions, let us introduce a set of traceless symmetric tensors of all ranks $C(x) = (c(x), \dots, c^{n(k)}(x), \dots)$,

$$\eta_{mn} c^{n(k+2)} = 0, \quad (2.1)$$

where $m, n, \dots = 0, \dots, d-1$ and η^{nm} is the mostly minus flat metric $\eta^{nm} = (1, -1, \dots, -1)$. In this paper we use the conventions of [11] convenient for the component analysis of complicated tensor structures: upper and lower indices denoted by the same letter should be first symmetrized (separately) and then the maximal possible number of lower and upper

indices should be contracted; the number of indices can be indicated in brackets by writing e.g. $n(k)$ instead of writing k times the index n . Underlined Latin indices are used for differential forms and vector fields in d -dimensional space-time with coordinates $x^{\underline{n}}$,

$$\underline{m}, \underline{n}, \dots = 0, \dots, d-1, \quad \partial_{\underline{n}} = \frac{\partial}{\partial x^{\underline{n}}}, \quad d = dx^{\underline{n}} \partial_{\underline{n}}. \quad (2.2)$$

Indices from the middle of the Latin Alphabet are fiber.

As shown in [2] the equations of motion of a free massless scalar field $c(x)$ in the flat d -dimensional space-time can be reformulated as the following infinite chain of equations

$$\partial_{\underline{n}} c_{n(k)} + e_{\underline{n}}^m c_{n(k)m} = 0, \quad (2.3)$$

where $e_{\underline{n}}^m$ is the flat space vielbein. Such a form of the dynamical equations expressing all the derivatives in terms of the fields was called “unfolded” in [12]. In the flat space one can choose $e_{\underline{n}}^m = \delta_{\underline{n}}^m$ and identify underlined (base) and non-underlined (fiber) indices.

The first two equations in (2.3) read

$$\partial_{\underline{n}} c = -c_{\underline{n}}, \quad (2.4)$$

$$\partial_{\underline{n}} c_{\underline{m}} = -c_{\underline{mn}}. \quad (2.5)$$

Eq. (2.4) tells us that $c_{\underline{n}}$ is the first derivative of c . Eq. (2.5) implies that $c_{\underline{nm}}$ is the second derivative of c . However, because of the tracelessness condition

$$c_n{}^n = 0, \quad (2.6)$$

it imposes the Klein-Gordon equation

$$\square c = 0. \quad (2.7)$$

The rest equations in (2.3) express highest tensors $c_{n(k)}$ in terms of the higher-order derivatives

$$c_{\underline{n}_1 \dots \underline{n}_k} = (-1)^k \partial_{\underline{n}_1} \dots \partial_{\underline{n}_k} c \quad (2.8)$$

imposing no further conditions on c . The tracelessness conditions (2.1) are all satisfied once the Klein-Gordon equation (2.7) is true.

Let T_k^p be a linear space of p -forms taking values in the space of rank- k totally symmetric traceless tensors. In other words, a general element of T_k^p is $c^{n(k)} = dx^{\underline{m}_1} \wedge \dots \wedge dx^{\underline{m}_p} c_{\underline{m}_1 \dots \underline{m}_p}{}^{n(k)}$, with $c_n{}^{n(k-1)} = 0$. Then $T^p = \sum_{k=0}^{\infty} \oplus T_k^p$ is the linear space of p -forms taking values in the space of all totally symmetric traceless tensors.

Let us introduce the operator

$$\sigma_- : T_k^p \rightarrow T_{k-1}^{p+1}, \quad (2.9)$$

$$(\sigma_- C)^{n(k-1)} = e_m \wedge c^{n(k-1)m}, \quad \sigma_-(T_0^p) = 0, \quad (2.10)$$

where e_n is the frame 1-form $e^n = dx^{\underline{n}} e_{\underline{n}}^n$. (In the flat space we can set $e^n = dx^{\underline{n}} \delta_{\underline{n}}^n = dx^n$.) The operator σ_- has the following important properties

$$(\sigma_-)^2 = 0, \quad \sigma_- d + d \sigma_- = 0. \quad (2.11)$$

The chain of equations (2.3) then takes the form

$$(d + \sigma_-)C = 0. \quad (2.12)$$

The compatibility condition $(d + \sigma_-)^2 = 0$ of this system holds as a consequence of (2.11).

Let us now address the question of the uniqueness of the equation (2.12) within the class of equations formulated in T^p in terms of the exterior differential and the frame 1-form. The only Lorentz covariant possibility is to write a chain of equations

$$\mathcal{D}C = 0, \quad (2.13)$$

$$\mathcal{D} = d + \sigma_- + \sigma_+, \quad (2.14)$$

with $\sigma_+ : T_k^p \rightarrow T_{k+1}^{p+1}$ being some operator of the form

$$(\sigma_+ C)^{n(k+1)} = f(k)P_\perp(e^n c^{n(k)}), \quad (2.15)$$

where $f(k)$ are some unknown coefficients and P_\perp is the projector to the subspace of traceless tensors, i.e.

$$(\sigma_+ C)^{n(k+1)} = f(k)\left(e^n c^{n(k)} - \frac{k}{d + 2(k-1)}\eta^{nn}e_m c^{n(k-1)m}\right). \quad (2.16)$$

The compatibility condition

$$\mathcal{D}^2 = 0 \quad (2.17)$$

requires in addition to (2.11)

$$\sigma_+ d + d\sigma_+ = 0, \quad (2.18)$$

$$\sigma_+ \sigma_+ = 0, \quad (2.19)$$

$$\{\sigma_+, \sigma_-\} = 0. \quad (2.20)$$

The first two conditions are trivially satisfied while the third imposes the following restrictions on the coefficients $f(k)$:

$$f(k) = \frac{(k+1)(d+2(k-1))}{k(d+2k)}f(k-1). \quad (2.21)$$

The generic solution of these equations is

$$f(k) = -m^2 \frac{k+1}{2k+d}, \quad (2.22)$$

where m^2 is an arbitrary constant.

The first two equations in the chain (2.13) give

$$\partial_{\underline{n}} c + e_{\underline{n}}^m c_m = 0, \quad (2.23)$$

$$\partial_{\underline{n}} c_n + e_{\underline{n}}^m c_{nm} - \frac{m^2}{d} e_{\underline{n}\underline{n}} c = 0. \quad (2.24)$$

Contracting indices in the second equation and substituting c_n from (2.23) we obtain

$$(\square + m^2)c = 0. \quad (2.25)$$

Therefore, the ambiguity in the coefficients $f(k)$ expresses the ambiguity in the parameter of mass in the Klein-Gordon equation reformulated in the form (2.13). An equivalent formulation for the massive scalar field in the flat space of an arbitrary dimension was given in [13]. Note that (2.8) has the same form for any value of m^2 .

3 “Unfolded” Scalar Field Equations in AdS_d

To generalize these results to AdS_d , consider the gauge fields $A_{\underline{n}}^{MN} = -A_{\underline{n}}^{NM}$ for the AdS_d algebra $o(d-1, 2)$, ($M, N = 0, \dots, d$). Setting $\omega_{\underline{n}}^{nm} = A_{\underline{n}}^{nm}$ and $e_{\underline{n}}^n = \lambda^{-1} A_{\underline{n}}^{nd}$, where λ is a constant, the $o(d-1, 2)$ —Yang-Mills strengths acquire the form

$$R_{\underline{nm}}^{nm} = \partial_{\underline{n}} \omega_{\underline{m}}^{nm} + \omega_{\underline{n}}^n{}_t \omega_{\underline{m}}^{tm} - \lambda^2 e_{\underline{n}}^n e_{\underline{m}}^m - (\underline{n} \leftrightarrow \underline{m}), \quad (3.1)$$

$$R_{\underline{nm}}^n = \lambda \partial_{\underline{n}} e_{\underline{m}}^n + \lambda \omega_{\underline{n}}^n{}_m e_{\underline{m}}^m - (\underline{n} \leftrightarrow \underline{m}). \quad (3.2)$$

The fields $e_{\underline{n}}^n$ and $\omega_{\underline{n}}^{nm}$ are identified with the vielbein and Lorentz connection. Provided that $\det|e_{\underline{n}}^n| \neq 0$, $\lambda^{-1} R_{\underline{nm}}^n$ and $R_{\underline{nm}}^{nm}$ identify, respectively, with the torsion tensor and Riemann curvature tensor (corrected by the λ -dependent “cosmological term”) in the vielbein formulation of gravity. The equations

$$R_{\underline{nm}}^{nm} = 0 \quad (3.3)$$

and

$$R_{\underline{nm}}^n = 0 \quad (3.4)$$

describe d -dimensional anti de-Sitter space of radius λ^{-1} .

The Lorentz covariant derivative is defined according to

$$D_{\underline{n}} f^{nm\dots} = \partial_{\underline{n}} f^{nm\dots} + \omega_{\underline{n}}^n{}_t f^{tm\dots} + \omega_{\underline{n}}^m{}_t f^{nt\dots} + \dots \quad (3.5)$$

From (3.1) and (3.3) it follows that in AdS_d

$$[D_{\underline{n}}, D_{\underline{m}}](f^{nm\dots}) = \lambda^2 (e_{\underline{n}}^n e_{\underline{m}t} f^{tm\dots} + e_{\underline{n}}^m e_{\underline{m}t} f^{nt\dots} + \dots) - (\underline{n} \leftrightarrow \underline{m}). \quad (3.6)$$

To generalize the scalar field equation in the form (2.13) to AdS_d case, we set

$$\mathcal{D} = D + \sigma_- + \sigma_+ \quad (3.7)$$

with $D = dx^{\underline{n}} D_{\underline{n}}$ and some new operator σ_+ . The compatibility condition $\mathcal{D}\mathcal{D} = 0$ in the AdS_d case requires the same properties of σ_- and σ_+ (2.11), (2.18), (2.19), but modifies (2.20) to

$$(\{\sigma_+, \sigma_-\}C)^{n(k)} = -(DD(C))^{n(k)} = -k\lambda^2 e^n \wedge e_m c^{n(k-1)m} \quad (3.8)$$

as a result of (3.6). Taking again σ_- and σ_+ in the form (2.10) and (2.15), respectively, (3.8) imposes the following conditions on $f(k)$

$$f(k) = \frac{(k+1)(d+2(k-1))}{k(d+2k)} (f(k-1) + k\lambda^2). \quad (3.9)$$

The generic solution of this equation is

$$f(k) = \frac{k+1}{2k+d} (\lambda^2 k(k+d-1) - m^2), \quad (3.10)$$

where m^2 is again an arbitrary parameter associated with the solution of the homogeneous part of (3.9). Analogously to the flat case, we derive from the first two equations of the chain (2.13) the massive Klein-Gordon equation³

$$(\square + m^2)c = 0, \quad \square = \eta_{nm}D^n D^m \quad (3.11)$$

and

$$c^n = -D^n c, \quad (3.12)$$

where $D^n = e^{\underline{n}} D_{\underline{n}}$ (with $e^{\underline{n}} e_{\underline{m}}^{\underline{n}} = \delta_{\underline{m}}^{\underline{n}}$). From the rest equations of the chain we obtain a covariantized version of (2.8)

$$c^{n_1 \dots n_k} = (-1)^k P_{\perp} D^{n_1} \dots D^{n_k} c. \quad (3.13)$$

Let us stress that the system (2.13) contains no restrictions on c beyond (3.11). One way to prove this is to analyze cohomology of the operator σ_- (see section 4).

If some consistent theory of fields of all spins possessing higher spin gauge symmetries exists, the operator \mathcal{D} should be interpreted as a result of linearization of full nonlinear higher spin covariant derivatives near the AdS_d vacuum solution with the background AdS_d gauge fields being of zero order. Other way around, one can take the form of the covariant derivative (3.7) as a starting point towards the full higher spin symmetry and its matter field representations. This strategy was proved to be successful in [14] for the analysis of $d = 4$ higher spin dynamics starting from the appropriate generalization of the covariant derivative (3.7) for the $d = 4$ higher spin massless fields. Also, the dynamical equations in the form (2.13) is a good starting point towards unfolded nonlinear higher spin equations (for more detail on this point we refer the reader to [2]).

4 σ_- Cohomology and Dynamical Equations

An interesting feature of the proposed formulation is that the equations of motion of a scalar field in the flat and AdS space admit a natural interpretation in terms of the cohomology group of σ_- . According to (2.9), σ_- increases a degree of differential forms decreasing a number of tensor indices and has the property $\sigma_-^2 = 0$. We show that the only nontrivial class $H^1(\sigma_-)$ of the first cohomology group of σ_-

$$T \xrightarrow{\sigma_-} T^1 \xrightarrow{H^1 \sigma_-} T^2, \quad (4.1)$$

belongs to T_1^1 and describes the left hand sides of the equations of motion for a scalar field. The constraints (2.8) or (3.13) fix particular representatives of the trivial cohomology classes.

Consider the restriction $DC|_{T_0^1}$ of DC to T_0^1 (i.e. the part in DC that does not carry vector indices). By definition of σ_- ,

$$\sigma_- (DC|_{T_0^1}) = 0. \quad (4.2)$$

³Let us note that there are several competing definitions of a massless scalar field in AdS_d with $m^2 = m_0^2 \neq 0$. For the massless field identified with the conformal scalar, $m_0^2 = -\frac{\lambda^2}{4}d(d-2)$.

The question is whether $DC|_{T_0^1} = \sigma_-(y)$ with some $y^n \in T_1^0$. In components, it is equivalent to $D_{\underline{n}}c = e_{\underline{nn}}y^n$. Obviously, the solution of this equation exists provided that the frame field $e_{\underline{nn}}$ is invertible. Thus $H_0^1 = 0$. Since y^n enters this equation just the same way as c^n enters (2.23) the fact that $H_0^1 = 0$ means that one can choose the field c^n in such a way that

$$((D + \sigma_-)C)|_{T_0^1} = 0. \quad (4.3)$$

Physically, this equation is interpreted as the constraint (2.23) expressing c^n via the first derivatives of the physical field c . Cohomologically, it fixes a representative of the trivial cohomology class c^n in terms of the derivatives of the dynamical field c . Note that this condition can be interpreted as a constraint because σ_- does not contain space-time derivatives.

Since $(\sigma_+C)|_{T_0^1} = 0$, the condition (4.3) is equivalent to

$$\mathcal{D}C|_{T_0^1} = 0. \quad (4.4)$$

Now let us show that $\mathcal{D}C|_{T_{k+1}^1}$ is σ_- -closed if

$$\mathcal{D}C|_{T_l^1} = 0 \quad \text{for } 0 \leq l \leq k. \quad (4.5)$$

Indeed, since the operators D and σ_+ map T_k to T_l with $l \geq k$ we obtain from (4.5) that

$$((D + \sigma_+)\mathcal{D}C)|_{T_l^1} = 0 \quad \text{for } l \leq k. \quad (4.6)$$

From $\mathcal{D}^2 = 0$ it follows then that

$$(\sigma_- \mathcal{D}C)|_{T_l^1} = 0 \quad \text{for } l \leq k \quad (4.7)$$

and, therefore,

$$\sigma_-(\mathcal{D}C|_{T_{k+1}^1}) = 0. \quad (4.8)$$

As a result, we conclude that

$$\mathcal{D}C|_{T_{k+1}^1} = \sigma_-(y_{k+2}) + h_{k+1}, \quad (4.9)$$

where $\sigma_-(y_{k+2})$ describes some σ_- -exact part (i.e. a trivial cohomology class) and h_{k+1} is a representative of a nontrivial cohomology H^1 . Obviously, the σ_- -exact part can be fixed to zero by adjusting a field $c^{n(k+2)}$ on the left hand side of (4.9). Therefore it is possible to impose constraints

$$\mathcal{D}C|_{T_{k+1}^1} = 0 \quad (4.10)$$

provided that the cohomology group is zero. If it is different from zero, the equation (4.10) imposes some differential equations on the bottom field $c(x)$.

Let us now find the cohomology group H^1 . A σ_- -closed 1-form $\hat{c}_{\underline{n}},^{n(k)}$ obeys

$$e_{\underline{nm}}\hat{c}_{\underline{m}},^{n(k-1)m} - e_{\underline{mm}}\hat{c}_{\underline{n}},^{n(k-1)m} = 0. \quad (4.11)$$

Therefore

$$e_{\underline{nm}}\hat{c}_{\underline{m}},^{n(k-1)m} = y_{\underline{nm}},^{n(k-1)}, \quad (4.12)$$

where $y_{\underline{nm}},^{n(k-1)}$ is symmetric in \underline{n} and \underline{m} , and symmetric and traceless in $n(k-1)$. Equivalently, using the fiber indices, we have

$$\hat{c}_{m,n(k)} = y_{mn,n(k-1)} \cdot \quad (4.13)$$

From this relation it follows that $y_{ml,n(k-1)}$ should be totally symmetric in all indices and traceless for $k \geq 2$. This implies that $\hat{c}_{\underline{n}},^{n(k)}$ is exact for all $k \neq 1$. Since $y_{\underline{nm}}$ is not required to be traceless there is a nontrivial cohomology class

$$\hat{c}_{\underline{n}},^n = \alpha e_{\underline{n}},^n \quad (4.14)$$

with an arbitrary parameter α . It is obvious that this element is σ_- -closed but not σ_- -exact because

$$e_{\underline{n}},^n \neq e_{\underline{n},m} X^{nm} \quad (4.15)$$

for any traceless X^{nm} . From the analysis of the sections 2 and 3 it is clear that sending this cohomology to zero in (4.9) is equivalent to imposing the Klein-Gordon equation. The fact that all other σ_- -cohomology groups in H^1 vanish means that the rest equations in the chain (2.13) and it's *AdS* generalization contain no further differential restrictions on the field $c(x)$, merely expressing higher components $c^{n(k)}$ in terms of derivatives of c .

To summarize, the equation (2.13) contains one differential equation, Klein-Gordon equation, along with an infinite set of constraints expressing the fields $c^{n(k)}$ $k \geq 1$ via derivatives of the physical field c according to (2.7) and (2.8) in the flat space or (3.11) and (3.13) in *AdS_d*.

Note that the dynamical field c represents a nontrivial cohomology group H^0 . Indeed, c is σ_- -closed but, being a 0-form, cannot be σ_- -exact. Let us stress that the physical field c should be σ_- -closed to make it possible to start the analysis of the equations by writing (4.2).

One of the lessons of this section is that, in order to have a chain of equations $\tilde{\mathcal{D}}\tilde{C} = 0$ which does not impose differential restrictions on a bottom field(s) merely expressing some of the 0-forms \tilde{C} in terms of derivatives of some other, one has to modify the setting in such a way that $\tilde{H}^1=0$. Such a problem setting is expected to be useful at the nonlinear level for the off-mass-shell formulation of the action principle compatible with the constraints written in the covariant form $\tilde{\mathcal{D}}\tilde{C} = 0$. For a free scalar field is developed in the section 8.

5 Free Action

In [11, 15] it has been shown that the “higher spin” covariant derivatives analogous to (3.7) can be used to build action functionals. Such a form of the higher spin action was used in [5] to introduce higher-spin-gravitational interactions at the cubic level. In this section we show how the free action for a scalar field can be formulated in terms of the covariant derivative (3.7). This action is expected to result from the linearization of some full higher spin invariant action describing higher spin and lower spin fields and formulated in terms of higher spin covariant derivatives. Hopefully, the proposed action will help to shed some light on the structure of a nonlinear higher spin action in any dimension and,

in particular, to build the full nonlinear action in $d = 4$ containing necessary lower spin matter fields.

We identify the fields c and c^n with the fields of the first-order Klein-Gordon action. Let us address the question whether there exists an action of the form

$$S^2 = \sum_{k=1}^{\infty} \frac{g(k)}{k!} \int_{M_d} e^{m_3} \wedge \dots \wedge e^{m_d} \wedge \epsilon_{m_1 \dots m_d} (\mathcal{D}C)^{m_1 n(k-1)} \wedge (\mathcal{D}C)^{m_2 n(k-1)}, \quad (5.1)$$

with some coefficients $g(k)$ (no symmetrization with respect to the indices m) that the equations of motion derived from this action are equivalent to the spin 0 equations in the form

$$(-D_n c^n + m^2)c = 0, \quad c^n = -D^n c. \quad (5.2)$$

In addition we require that

$$\frac{\delta S^2}{\delta c^{n(k)}} \equiv 0 \quad \text{for } k \geq 2. \quad (5.3)$$

The latter condition admits a natural interpretation in view of the formula (3.13) as the requirement that the free action does not contain higher derivatives of the scalar field c .

To simplify formulae, let us introduce the operator $E: T_k^p \rightarrow T_k^{d+p-2}$ and $N: T_k^p \rightarrow T_k^p$,

$$(EC)^{n(k)} = g(k) e^{m_1} \wedge \dots \wedge e^{m_{d-2}} \wedge c^{n(k-1)t} \epsilon_t{}^n{}_{m_1 \dots m_{d-2}}, \quad k \geq 1; \quad E(c) = 0, \quad (5.4)$$

$$(NC)^{n(k)} = k c^{n(k)} \quad (5.5)$$

for any p-form $c^{n(k)}$. The following elementary identities take place

$$\sigma_+ N = (N - 1)\sigma_+, \quad \sigma_- N = (N + 1)\sigma_-, \quad (5.6)$$

$$E\sigma_- = (-1)^d \frac{N+1}{N+d-1} \frac{g(N)}{g(N+1)} \sigma_- E, \quad N \geq 1, \quad (5.7)$$

$$E\sigma_+ = (-1)^d \frac{N+d-2}{N} \frac{g(N)}{g(N-1)} \sigma_+ E, \quad N \geq 2. \quad (5.8)$$

Let us introduce notation

$$(C^p, B^q) = \sum_k \frac{1}{k!} c^{n(k)} \wedge b_{n(k)}, \quad (5.9)$$

where $C^p \in T^p$ and $B^q \in T^q$. The following identities are true

$$(C^p, B^q) = (-1)^{pq} (B^q, C^p), \quad (5.10)$$

$$(C^p, EB^q) = (-1)^{pd-1} (EC^p, B^q), \quad (5.11)$$

$$(C^p, \sigma_+ B^q) = (-1)^p \left(\frac{f(N)}{N+1} \sigma_- C^p, B^q \right), \quad (5.12)$$

$$(C^p, \sigma_- B^q) = (-1)^p \left(\frac{N}{f(N-1)} \sigma_+ C^p, B^q \right). \quad (5.13)$$

With this notation, the action (5.1) reads

$$S^2 = \int_{M_d} (EDC, DC). \quad (5.14)$$

Since the form (5.9) is Lorentz invariant, the Stokes theorem can be written in the form

$$\int_{M_d} (C^p, DB^{d-p-1}) = -(-1)^p \int_{M_d} (DC^p, B^{d-p-1}) + (-1)^p \int_{\partial M_d} (C^p, B^{d-p-1}). \quad (5.15)$$

A local variation of the action is

$$\delta S^2 = \int_{M_d} (EDC, \mathcal{D}\delta C) + \int_{M_d} (E\mathcal{D}\delta C, DC) = 2 \int_{M_d} (EDC, \mathcal{D}\delta C). \quad (5.16)$$

Integrating by parts and using the definition (3.7) of the operator \mathcal{D} along with the identities (5.7), (5.8), (5.12) and (5.13) we obtain

$$\begin{aligned} \delta S^2 = (-1)^{d-1} 2 \int_{M_d} & \left(\left[\left[\frac{N+d-2}{N} \frac{g(N)}{g(N-1)} + \frac{N}{f(N-1)} \right] \sigma_+ + \right. \\ & \left. + \left[\frac{N+1}{N+d-1} \frac{g(N)}{g(N+1)} + \frac{f(N)}{N+1} \right] \sigma_- \right] EDC, \delta C \right), \end{aligned} \quad (5.17)$$

provided that $\delta C = N(N-1)\Upsilon$ for arbitrary Υ (i.e. δC does not contain δc and δc^n). The condition that this variation vanishes is equivalent to (5.3). Requiring the coefficients in front of σ_- and σ_+ to vanish we obtain the set of equations on $g(k)$

$$g(k) = -\frac{g(k-1)}{f(k-1)} \frac{k^2}{k+d-2}, \quad (5.18)$$

which admits a unique solution up to an arbitrary normalization factor that can be fixed as

$$g(1) = \frac{1}{2} \frac{d^2}{m^2}, \quad (5.19)$$

to have the standard normalization of the free scalar field action.

Substituting $f(k)$ (3.10) we obtain for the case of an arbitrary mass⁴ in AdS_d space

$$g(k) = \frac{(-1)^{k-1}}{2} \frac{1}{m^2 \lambda^{2k-2}} \frac{\Gamma(\alpha^+ + 2) \Gamma(\alpha^- + 2)}{\Gamma(k + \alpha^+ + 1) \Gamma(k + \alpha^- + 1)} \frac{d!}{(d-2)!!} \frac{(2k+d-2)!!}{(k+d-2)!} k!, \quad (5.20)$$

where

$$\alpha^\pm = \frac{1}{2} \left(d-3 \pm \sqrt{(d-1)^2 + 4 \frac{m^2}{\lambda^2}} \right). \quad (5.21)$$

⁴For the values of mass $m^2 = \lambda^2 k_0(k_0 + d - 1)$, $k_0 = 0, 1, \dots$ the function $g(k)$ tends to infinity when $k > k_0$ thus making the formula for the action (5.1) inapplicable. These special values of the parameter of mass are analogous to those found previously in [16] at the level of equations of motion for $d = 3$. Their appearance signals that for these special values of the parameter the infinite chains start not with the scalar c , but rather with the rank k_0 tensor $c^{n(k_0)}$ so that after appropriate redefinition of the normalization constant the action (5.1) turns out to be well-defined. We hope to consider this interesting question elsewhere.

For the flat case with nonzero mass we have

$$g(k) = \frac{(-1)^{k-1}}{2} \frac{1}{m^{2k}} \frac{d!}{(d-2)!!} \frac{(2k+d-2)!!}{(k+d-2)!} k!. \quad (5.22)$$

Note that $g(k)$ is such that, by virtue of (5.7) and (5.12), σ_- is conjugated to σ_+ with respect to the form $(E(C), B)$ restricted to T_k with $k \geq 2$, i.e.

$$(EC^p, \sigma_+ B^q) = (-1)^{p-1} (E\sigma_- C^p, B^q) \quad \text{for } C^p = N(N-1)\Upsilon^p. \quad (5.23)$$

With the help of this property it is trivial to see that $\delta S^2 = 0$ provided that $N(N-1)\delta C = 0$. This explains why we have obtained only one equation (5.18) from the requirement that the two coefficients vanish in (5.17).

Since the conjugation relation (5.23) does not take place for T_0^0 and T_1^0 , the equations of motion for the fields c and c^n turn out to be nontrivial having a form

$$\frac{\delta S^2}{\delta c} = 2(-1)^d \frac{m^2}{d} \sigma_- E(\mathcal{D}C|_{T_1^1}) = 0, \quad (5.24)$$

$$\frac{\delta S^2}{\delta c^1} = -2E\sigma_+(\mathcal{D}C|_{T_0^1}) = 0. \quad (5.25)$$

It is elementary to see that these equations are equivalent to the free field equations (5.2). In fact, the equation (5.25) is equivalent to

$$(\mathcal{D}C)|_{T_0^1} = 0 \quad (5.26)$$

(i.e., $\text{Ker}(E\sigma_+)|_{T_0^1} = 0$), while the equation (5.24) is equivalent to

$$(\mathcal{D}C)|_{T_1^1} = \sigma_- X \quad (5.27)$$

with an arbitrary X ($\text{Ker}(\sigma_- E)|_{T_1^1} = \sigma_- X$). One can always adjust such a field c^{nm} that $X = 0$. This condition can be interpreted as some constraint on c^{nm} . As shown in the sections 2, 3 and 4, by imposing constraints on the higher fields $c^{n(k)}$ one can achieve $\mathcal{D}C = 0$ provided that the dynamical equations (5.24) and (5.25) are satisfied.

Although the bilinear action is independent of (local variations of) the “extra fields” $c^{n(k)}$ with $k \geq 2$, it is useful to express “extra fields” $c^{n(k)}$ with $k \geq 2$ in terms of the derivatives of the dynamical scalar field c because extra fields contribute at the interaction level and have to be taken into account in the boundary terms. The natural choice for these constraints is according to (3.13). At the action level this can be achieved by adding the term

$$S^c = \sum_{k=2} \int_{\Omega} \gamma^{n_1 \dots n_k} (c_{n_1 \dots n_k} - (-1)^k D_{n_1} \dots D_{n_k} c), \quad (5.28)$$

with Lagrange multipliers $\Gamma = (\gamma^{n(2)}, \gamma^{n(3)} \dots)$, ($\gamma^{n(k)}$ is symmetric and traceless). The action

$$S = S^2 + S^c \quad (5.29)$$

leads to the equations (5.24) and (5.25) along with

$$c_{n_1 \dots n_k} = (-1)^k P_{\perp} D_{n_1} \dots D_{n_k} c, \quad \gamma^{n(k)} = 0, \quad k \geq 2. \quad (5.30)$$

Taking into account the results of the section 4, these equations are equivalent to

$$\mathcal{D}C = 0, \quad (5.31)$$

$$\Gamma = 0. \quad (5.32)$$

Let us note that it is possible to extend the set of Lagrangian multipliers by adding γ^n and γ . The scalar γ cancels out from the action. The term with γ^n contributes but leads to the equivalent dynamics with $\gamma^n = 0$ as a consequence of the equations of motion.

So far, we have considered local variations in the bulk. If ∂M_d is nontrivial, the boundary terms have to be taken into account. It is elementary to see directly that

$$\begin{aligned} S^2 = \int_{M_d} & \left(2(E\sigma_+c, Dc^n) + (E\sigma_+c, \sigma_+c) - (E\sigma_+\sigma_-c^n, c^n) \right) + \\ & + (-1)^{d-1} \int_{\partial M_d} \left((E\mathcal{D}C, C) - (E\sigma_+c, c^n) \right). \end{aligned} \quad (5.33)$$

Thus, the action S^2 equals to the standard first-order Klein-Gordon action modulo boundary terms which have to be subtracted to make the two actions equivalent.

6 Fock Space Notation

Instead of working with infinite sets of tensors C it is convenient to use the Fock-space language analogous to that used in [15] for spin $s \geq 1$ fields. Namely, we introduce the creation and annihilation operators a_n and a_m^+ obeying the commutation relations

$$[a_n, a_m^+] = \eta_{nm}, \quad (6.1)$$

$$[a_n, a_m] = [a_n^+, a_m^+] = 0. \quad (6.2)$$

Given set of p-forms $c^{n(k)}$, we introduce a Fock vector

$$|c(p)\rangle = \sum_{k=0}^{\infty} |c(p, k)\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} a_{n_1}^+ \cdots a_{n_k}^+ c^{n_1 \dots n_k} |0\rangle. \quad (6.3)$$

It is convenient to introduce operators

$$N^{++} = \frac{1}{2} a_n^+ a^{+n}, \quad N^{--} = \frac{1}{2} a_n a^n \quad (6.4)$$

and

$$N = a_n^+ a^n, \quad (6.5)$$

satisfying the commutation relations

$$[N, N^{++}] = 2N^{++}, \quad [N, N^{--}] = -2N^{--}, \quad (6.6)$$

$$[N^{--}, N^{++}] = N + \frac{d}{2}, \quad (6.7)$$

which transform to the sl_2 commutation relations by the trivial shift $N^0 = N + \frac{d}{2}$.

The subspace of traceless symmetric tensors is extracted by the condition

$$N^{--}|c(p, k)\rangle = 0. \quad (6.8)$$

The explicit form of the projector P_\perp to the traceless tensors is complicated. For practical computations it is however enough to use the following simple formulae

$$a_n^+ P_\perp = P_\perp a_n^+ + (N + \frac{d}{2} - 2)^{-1} N^{++} a_n P_\perp, \quad (6.9)$$

$$a_n P_\perp = P_\perp a_n + (N + \frac{d}{2} - 1)^{-1} P_\perp a_n^+ N^{--}. \quad (6.10)$$

We have

$$\sigma_- = e_n a^n, \quad (6.11)$$

$$\sigma_+ = \frac{f(N-1)}{N} P_\perp e^n a_n^+ = \frac{f(N-1)}{N} \left(e^n a_n^+ - (N + \frac{d}{2} - 2)^{-1} N^{++} \sigma_- \right), \quad (6.12)$$

$$D^n = e^{\underline{n}n} (\partial_{\underline{n}} + \omega_{\underline{n}}^t{}_m a_t^+ a^m), \quad (6.13)$$

$$E = \frac{g(N)}{N} e^{t_1} \wedge \dots \wedge e^{t_{d-2}} \epsilon_m{}^n{}_{t_1 \dots t_{d-2}} a_n^+ a^m. \quad (6.14)$$

It is also useful to use Lorentz invariant “base” oscillators

$$a_{\underline{n}} = e_{\underline{n}}{}^n a_n, \quad a_{\underline{n}}^+ = e_{\underline{n}}{}^n a_n^+, \quad a^{\underline{n}} = e^{\underline{n}n} a_n, \quad a^{+\underline{n}} = e^{\underline{n}n} a_n^+, \quad (6.15)$$

satisfying

$$D(a_{\underline{n}}) = 0, \quad D(a^{\underline{n}}) = 0, \quad D(a_{\underline{n}}^+) = 0, \quad D(a^{+\underline{n}}) = 0 \quad (6.16)$$

by virtue of the zero torsion condition (3.4). Using the definition (2.14) of \mathcal{D} and introducing

$$D^+ = D^n a_n^+ = D_{\underline{n}} a^{+\underline{n}} \quad (6.17)$$

we rewrite the action (5.29) in the form

$$S = (-1)^{d-1} \langle \mathcal{D}C | E | \mathcal{D}C \rangle + \langle \Gamma | C \rangle - \langle \Gamma | \exp(-D^+) c | 0 \rangle. \quad (6.18)$$

7 Yang-Mills Interaction

Although the free action (5.14) was shown to be equivalent modulo some boundary terms to the usual first-order Klein-Gordon action, the minimal Yang-Mills interaction introduced via covariant derivatives leads to different results in the two cases. This is not surprising because in the equivalence proof we have used the fact that $\mathcal{D}^2 = 0$ for the background AdS geometry. This is no longer true in the presence of the Yang-Mills connection A

$$\mathcal{D} \rightarrow \mathcal{D}^{YM} = \mathcal{D} + A, \quad (7.1)$$

with

$$(\mathcal{D}^{YM})^2 = G, \quad G = dA + A \wedge A. \quad (7.2)$$

Here A and G take values in some representation t of the gauge group g , in which the scalar fields $c^{n(k)}$ take their values (i.e. $c^{n(k)} \rightarrow c^{\alpha}_{n(k)}$, $A \rightarrow A^{\alpha}_{\beta}$). The two actions can therefore differ by some terms proportional to the Yang-Mills field strength G .

An interesting consequence of the reformulation of the scalar field action in the form (5.14) is that it immediately leads to the pseudolocally exact form of the conserved spin-1 current generalizing the $d=3$ results of [8] for spin -1 currents to the case of scalar matter field of arbitrary mass in AdS_d or nonzero mass in the flat space. Indeed, the action (5.14) with the minimal Yang-Mills interaction reads

$$S^{gau} = \int_{M_d} tr \left(E \mathcal{D}^{YM} C, \mathcal{D}^{YM} C \right) . \quad (7.3)$$

The spin-1 conserved current is

$$J^{\underline{n}}_{\alpha}{}^{\beta} = \frac{\delta}{\delta A_{\underline{n}}^{\alpha}{}_{\beta}} S^{gau} |_{A=0} . \quad (7.4)$$

For the action (7.3) we have

$$\delta S^{gau} |_{A=0} = \int_{M_d} \left((E \delta A(C), \mathcal{D} C) + (E \mathcal{D} C, \delta A(C)) \right) = 2 \int_{M_d} (E \mathcal{D} C, \delta A(C)) \quad (7.5)$$

Taking into account that the free equations of motion (5.24), (5.25) along with the constraints (5.30) imply $\mathcal{D} C = 0$ we arrive at the paradoxical result that the Yang-Mills current derived from the action (5.14) vanishes on-mass-shell despite the fact that the action is explicitly invariant under the Yang-Mills symmetry⁵. An important related comment is that the proposed formulation is applicable just in those cases when either the S -matrix cannot be defined (AdS case) or the contribution of the corresponding three-particle vertices to the scattering amplitude vanishes by kinematical reasons (flat space case with $m \neq 0$).

Usually one argues that any terms in the current interaction that vanish on-mass-shell are irrelevant because one can compensate them by some local field redefinition

$$C \rightarrow C' = C + AC^2 . \quad (7.6)$$

In the case under consideration one has to take into account however that the variation contains infinitely many terms with all derivatives of the scalar field c via the highest components $c^{n(k)}$ (3.13). Therefore, a field redefinition (7.6) compensating interactions in the action (5.14) may be nonlocal. Such expressions containing infinite series in powers of derivatives were called in [8] pseudolocal. It is not surprising that some interaction can be compensated by a nonlocal field redefinition (for example with the aid of the Green function). The naive conclusion that the action (7.3) does not describe nontrivial interactions is therefore wrong.

⁵Let us note that since the variation of the free action with respect to the extra fields vanishes identically, it is enough for the analysis of the cubic interaction to use the free constraints (3.13). In other words, corrections due to the Yang-Mills covariantization of the expression (5.28) may only affect quartic interactions. Note that the term with Lagrange multipliers does not affect this consideration either because $\Gamma = 0$ on-mass-shell. Alternatively, one can impose the constraints by hand without introducing Lagrange multipliers.

The lesson is that usual current interactions playing a fundamental role in the local field theory may not be that important in theories with interactions containing infinite series in higher derivatives. This is true both for massive modes in the flat space (like in the string theory) and for theories with arbitrary mass in the AdS background like in the higher spin theories. One way to see this is to show that ordinary Yang-Mills current is on-mass-shell exact in the class of pseudolocal expansions. To this end, let us compare the action (5.14) with the standard first-order Klein-Gordon action with the Yang-Mills interaction. Proceeding as in the section 5 we arrive at the following result

$$S^{gau} = \int_{M_d} \left(2 \left(E\sigma_+ c, D^{YM} c^n \right) + (E\sigma_+ c, \sigma_+ c) - (E\sigma_+ \sigma_- c^n, c^n) \right) + \int_{M_d} (EGC, C) \\ + (-1)^{d-1} \int_{\partial M_d} \left((A \mathcal{D}^{YM} C, C) - (E\sigma_+ c, c^n) \right), \quad (7.7)$$

where $D^{YM} = D + A$. The first term in this formula just describes the covariantized Klein-Gordon first-order action S_{KG}^{gau} . Therefore, we obtain that the difference

$$\Delta = S^{gau} - S_{KG}^{gau} \quad (7.8)$$

is proportional to the Yang-Mills field strength up to some boundary terms

$$\Delta = \int_{M_d} (EGC, C) + \text{boundary terms}. \quad (7.9)$$

From this formula one immediately derives the pseudolocally exact representation for the spin-1 current. Indeed,

$$S_{KG}^{gau} = S_{KG}^{free} + \int_{M_d} \text{tr}(*J \wedge A) + O(A^2). \quad (7.10)$$

On the other hand we have

$$\frac{\delta S^{gau}}{\delta A_{\underline{n}}^\alpha{}_\beta} \Big|_{A=0} \sim 0, \quad (7.11)$$

where the \sim implies “on-mass-shell” equality and

$$\Delta = \int_{M_d} \text{tr}(U \wedge G), \quad U = (EC, C). \quad (7.12)$$

Taking into account (7.8) we obtain

$$*J \sim (-1)^d dU \quad (7.13)$$

with U (7.12) being an infinite expansion in higher derivatives due to (5.30).

An interesting problem for the future is to study the role of the boundary terms in (7.7) in the context of the AdS/CFT correspondence [17] to clarify to which extend the dynamics of the bulk action (7.7) is encoded in the boundary actions.

8 Off-Mass-Shell System

The approach developed in this paper is analogous to that developed in [15] for massless gauge fields in any dimensions and in [6] for all massless fields in $d=4$. It is adequate for the analysis of the nonlinear equations of motion as some deformation of (5.31). Also it is useful for the analysis of the interactions of higher spin gauge fields at the cubic level because the analysis of the Noether current interactions is essentially on-mass-shell⁶ so that constraints and equations of motion can be used simultaneously in the form analogous to (5.31). Beyond the cubic level one needs however appropriately modified off-mass-shell constraints compatible with the higher spin symmetries. In [11] the problem of formulation of invariant constraints was called the “extra field” problem. Here we undertake a step in this direction solving the problem of separation of constraints and dynamical field equations for the simplest case of a free scalar field.

The idea is to generalize the approach of the section 4 in such a way that the generalized covariant constancy conditions

$$\tilde{\mathcal{D}}\tilde{C} = 0 \quad (8.1)$$

for some extended set of the fields \tilde{C} express all the fields in terms of derivatives of the dynamical scalar field $c(x)$ imposing no differential restrictions on the latter. As shown in the section 4, nontrivial differential equations on the physical field c are associated with the cohomology of the operator σ_- . The idea therefore is to look for such a modified covariant derivative $\tilde{\mathcal{D}}\tilde{C}$ ($\tilde{\mathcal{D}}^2 = 0$) that the $\tilde{\sigma}_-$ -cohomology of $\tilde{\mathcal{D}}$ is trivial. An additional requirement is that together with the Klein-Gordon equation for c , the equation (8.1) should be equivalent to the system (3.11), (3.13). These conditions can be achieved by extending the set of tensors C to all symmetric but not necessarily traceless tensors $\tilde{C} = (\tilde{c}, \dots, \tilde{c}^{n(k)}, \dots)$.

Let \tilde{T}_k^p be a linear space of p -forms taking values in the space of rank- k totally symmetric tensors and $\tilde{T}^p = \sum_{k=0}^{\infty} \oplus \tilde{T}_k^p$, $\tilde{T} = \sum_{p=0}^{\infty} \oplus \tilde{T}^p$. The space \tilde{T}^p can be realized as a Fock space of the section 6 relaxing the tracelessness condition (6.8).

Let the operator $\tilde{\sigma}_- \tilde{T}_k^p \rightarrow \tilde{T}_{k-1}^{p+1}$ be defined as before

$$\tilde{\sigma}_- = e^n a_n \quad (8.2)$$

(equivalently, in terms of components, $(\tilde{\sigma}_- \tilde{C})^{n(k-1)} = e_m \wedge \tilde{c}^{n(k-1)m}$). Obviously,

$$\tilde{\sigma}_- \tilde{\sigma}_- = 0, \quad (8.3)$$

$$D\tilde{\sigma}_- + \tilde{\sigma}_- D = 0. \quad (8.4)$$

Note that the operator $\tilde{\sigma}_-$ is different from σ_- because it acts in a different space. This is manifested by the fact that, as is necessary for our construction, the first $\tilde{\sigma}_-$ -cohomology group \tilde{H}^1 is trivial. (The explicit proof of this fact is not given here because it is a simplified version of that for the σ_- -cohomology in the section 4.) On the other hand, $\tilde{\sigma}_-$

⁶Note that the situation with higher spin gauge fields considered in [5] is different from the one with the scalar field discussed in this paper in that respect that, for any three fixed spins, only a finite number of terms that vanish on-mass-shell appear in the gauge variation of the higher spin action of Ref.[5] and, therefore, the fact that there exists some deformed gauge transformation that leaves the action invariant [5] is the well-defined local statement.

leaves invariant the subspace $T \in \tilde{T}$ spanned by traceless tensors and σ_- is the restriction of $\tilde{\sigma}_-$ to T

$$\tilde{\sigma}_-|_T = \sigma_- . \quad (8.5)$$

Let us look for the operator $\tilde{\mathcal{D}}$ in the form

$$\tilde{\mathcal{D}} = D + \tilde{\sigma}_- + \tilde{\sigma}_+ , \quad (8.6)$$

where $\tilde{\sigma}_+ : \tilde{T}_k^p \rightarrow \tilde{T}_{k+1}^{p+1}$ is some operator demanded to satisfy the conditions

$$\tilde{\sigma}_+ \tilde{\sigma}_+ = 0 , \quad (8.7)$$

$$D \tilde{\sigma}_+ + \tilde{\sigma}_+ D = 0 , \quad (8.8)$$

$$\{\tilde{\sigma}_+ , \tilde{\sigma}_-\} = -DD = -\lambda^2 e^n a_n^+ e^m a_m \quad (8.9)$$

to guarantee the compatibility condition

$$\tilde{\mathcal{D}} \tilde{\mathcal{D}} = 0 . \quad (8.10)$$

In addition it is convenient to require that

$$\tilde{\sigma}_+|_T = \sigma_+ \quad (8.11)$$

to interpret the on-mass-shell chain $\mathcal{D}C = 0$ as the restriction of (8.1) to T .

Let us look for the operator $\tilde{\sigma}_+$ in the form

$$\tilde{\sigma}_+ = p(N) e^n a_n^+ + q(N) N^{++} e^n a_n \quad (8.12)$$

with some coefficients $p(N)$ and $q(N)$. The conditions (8.9) and (8.7) give rise to the following equations

$$q(N+1) = p(N+1) - p(N) + \lambda^2 \quad (8.13)$$

and

$$p(N)q(N-1) - q(N)p(N-1) + q(N)q(N-1) = 0 , \quad (8.14)$$

respectively. The condition (8.11) is equivalent to the requirement that $N^{--} \tilde{\sigma}_+ = X_+ N^{--}$ with some operator X_+ . By virtue of (6.7) it gives

$$p(N) = -(N + \frac{d}{2} - 2)q(N) . \quad (8.15)$$

The generic solution of all these conditions reads

$$p(N) = \frac{1}{2} \left(\lambda^2 (N + \frac{d}{2} - 2) + \frac{m_c^2}{N + \frac{d}{2} - 1} \right) , \quad (8.16)$$

where m_c^2 is an arbitrary parameter. When $\tilde{\sigma}_+$ is applied to a traceless vector it reproduces the operator σ_+ (6.12), (3.9) of the on-mass-shell problem with

$$m_c^2 = m^2 + \frac{\lambda^2}{4} d(d-2) . \quad (8.17)$$

Note that $m_c^2 = 0$ corresponds to the conformal case (see footnote 3).

To summarize, we have shown that on-mass-shell covariant derivative (3.7) admits such a generalization to a larger set of fields that the covariant constancy conditions (8.1) do not impose any dynamical equations on the matter field $c(x)$ merely expressing higher components in the set \tilde{C} via higher derivatives of $c(x)$. Because the operator $\tilde{\mathcal{D}}$ is defined in such a way that it reduces to \mathcal{D} when restricted to the subspace $T^0 \subset \tilde{T}^0$, the dynamical field equations turn out to be equivalent to the condition that all fields in \tilde{T}^0/T^0 vanish. It is an interesting problem for the future to find an action principle leading to such field equations.

9 Conclusions

In this paper the dynamics of a scalar field in AdS_d is formulated in terms of certain “higher spin” covariant derivatives both at the level of equations of motion and at the Lagrangian level.

Interestingly enough the proposed formalism leads to the interpretation of the dynamical field equations (i.e. Klein-Gordon equation) as the requirement that the fields belong to the trivial class of certain cohomology group, σ_- -cohomology. An interesting problem for the future is to extend this interpretation to other types of relativistic fields and to clarify its group-theoretical meaning.

The new action principle for a scalar field in arbitrary dimension proposed in this paper is shown to be equivalent (modulo boundary terms) to the standard first-order Klein-Gordon action at the free field level but different at the interaction level leading to pseudolocal interactions containing derivatives of all orders. This action is defined for massive fields in the flat space and for fields of an arbitrary mass (requiring some redefinition for special values of $\frac{m^2}{\lambda^2}$ — see footnote 4) in AdS_d . It contains inverse powers of either the parameter of mass or the cosmological constant in front of the terms with higher derivatives. In this sense it is analogous to the higher spin actions constructed previously in [11] and to the actions for massive higher spin fields constructed recently in [9, 10]. This picture fits nicely the superstring picture in which interactions contain powers of the parameter α' that fixes (inverse) mass scale in the theory.

An interesting conclusion of this paper is that current interactions do not play a fundamental role in the theories admitting infinite expansions in higher derivatives at the interaction level like higher spin theories and string theories. This conclusion is in fact welcome for any theory expected to be identified with one or another phase of the string theory because ordinary local field theory Feynman diagram expansion (i.e., with local current vertices) contradicts duality in the string theory [18]. The scalar field action presented in this paper illustrates how the ordinary field-theoretical actions reformulated in the higher spin inspired way can escape this potential conflict. An important related point is that the proposed formulation is applicable just in those cases when either the S -matrix cannot be defined (AdS case) or the contribution of the corresponding three-particle vertices to the scattering amplitude vanishes by kinematical reasons (flat space case with $m \neq 0$).

The authors are grateful to S.Prokushkin for collaboration at the early stage of the work and also to R.Metsaev and I.Tipunin for useful discussions. This research was supported in part by INTAS, Grant No.96-0538 and by the RFBR Grant No.99-02-16207.

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